Leximax and leximin extension rules for ranking sets as final outcomes with null alternatives

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Abstract

The objective of this study is to characterize the *leximax* and *leximin* extension rules for ranking sets as final outcomes. To rank any two subsets, we introduce *null* alternatives. We assume that each null alternative indicates 'choosing not to choose each alternative.' By adding null alternatives into each subset in which existing alternatives are not included, the cardinality of each transformed subset becomes equal to one of the set of all alternatives. Furthermore, all (null) alternatives in every transformed subset are rearranged in descending order. From these operations, we can rank all subsets lexicographically. The major result is axiomatization of the *leximax* and *leximin* extension rules by *dominance* axioms. Additionally, we clarify that the *leximax* and *leximin* extension rules satisfy *monotonicity* and *extended independence*.

1 Introduction

Three classes have been proposed in theories of *ranking sets of alternatives* (or *objects*) (see [2]). The first class is *complete uncertainty* (see [1, 3, 7, 10, 15, 17, 22]). In this class, probabilities are given for all alternatives in each subset, and each agent receives only one alternative chosen at random from a subset. The second class is *opportunity sets* (see [5, 6, 9, 11, 13, 14, 18, 19, 23]). In this class, each agent can choose only one alternative from a subset. The third class is *sets as final outcomes* (see [4, 12, 16, 20]). In this study, we analyse individual preference extension rules, not collective ones, for ranking sets as final outcomes.

From decision making in daily lives to policy making, we often choose a set of several alternatives based on a preference order of alternatives. Furthermore, we generally consider 'compatibility' of alternatives. For example, suppose that a soccer coach wants to hire two players, and there are three candidates: two forwards (FW_A and FW_B), and one defender (DF). Additionally, for the coach, assume that a preference order is FW_A, FW_B, and DF, and FW_A and FW_B are incompatible. In this case, the coach will hire FW_A and DF. However, in order to discuss how any two alternatives are (in)compatible, we need a reference point. We thus argue the simplest case with no compatibility of alternatives as the reference point. We then assume the following situation: an agent tries to plate some fruits at a reception. Note that the agent is assumed not to use a blender for consuming the fruits. Now, the goal is to lexicographically rank all subsets of the set of fruits. Particularly, we characterize the *leximax* and *leximin* extension rules.[1]

The main reference for this study is Bossert [4]. He characterized a group of lexicographic extension rules, including the leximax, leximin, median-based, and other lexicographical rules. However, subsets can be ranked if and only if they have the same cardinality, because

[1]We do not characterize a *median-based* rule, such as the Nitzan and Pattanaik [16] rule, because it belongs to a subgroup, including rules such as the *maximax* and *maximin* ones. Even if a *leximedian* rule can be defined, it would differ from the leximax and leximin rules, because we need one additional rule to rank pairs of both outsides of medians. Thus, it is excluded from this study.

Bossert [4] assumed a specific situation such as many-to-one matching with a given quota. To remove the restriction, we can use *empty slots*, which were introduced in Roth and Sotomayor [21]. Each empty slot is similar to an empty set or outside option, and assumed to be added to each subset whose cardinality is smaller than a given quota. However, we consider situations in which there is no fixed quota to apply extension rules to more general choice theories.

We then introduce *null* alternatives, which are assumed to indicate 'choosing not to choose alternatives.' If each alternative is not included in each subset, we add its null alternative to the subset, and rearrange all (null) alternatives in descending order. Following these operations, all subsets can be ranked lexicographically. Additionally, each alternative is defined as *(un)desirable* if and only if it is strictly better (worse) than its null alternative, and *neutral* if and only if it and its null alternative are indifferent. For example, suppose that $\{a, b, c, d\}$ is the set of all alternatives, and for an agent, a and b are indifferent, b is desirable, all null alternatives are indifferent, c is undesirable, and c is strictly better than d. Additionally, assume that n_a denotes a null alternative of a. If the agent forcibly ranks $\{a, c, d\}$ and $\{b, d\}$ based on the leximax criteria, $\{a, c, d\}$ is strictly better than $\{b, d\}$. However, this is non-intuitive because the third and second alternatives of $\{a, c, d\}$ and $\{b, d\}$ are the same, and c is undesirable. By adding null alternatives and rearranging them, $\{a, c, d\}$ and $\{b, d\}$ are transformed into $\{a, n_b, c, d\}$ and $\{b, n_a, n_c, d\}$, respectively. We thus obtain that $\{b, d\}$ is strictly better than $\{a, c, d\}$.

In this study, we do not assume a property called *null-indifference*. This requires that all null alternatives are indifferent, such as empty slots. The following example illustrates an advantage in relaxing *null-indifference*. Suppose that $\{a, b, c\}$ is the set of all alternatives, and an agent strictly prefers a to b, likes a and b intermediately, and hates c enormously. If we assume *null-indifference*, $\{a, c\}$ and $\{b\}$ will be transformed into $\{a, n_b, c\}$ and $\{b, n_a, n_c\}$, respectively, and the agent will strictly prefer $\{a, c\}$ to $\{b\}$ based on the leximax criteria. However, this is non-intuitive because the agent hates c enormously. Now, if the agent is allowed to strictly prefer n_c to a, the two subsets will be transformed into $\{a, n_b, c\}$ and $\{n_c, b, n_a\}$, and the agent will strictly prefer $\{b\}$ to $\{a, c\}$. Note that we need a weaker property than *null-indifference* to make a preference order of alternatives equivalent to one of their singleton sets. We thus assume *asymmetry of desirability*. This requires that a preference order of any two alternatives and one of their null alternatives are opposite.

The major result is characterization of the *leximax* and *leximin* extension rules by *dominance* axioms. Furthermore, we obtain that the *leximax* and *leximin* extension rules satisfy *monotonicity* and *extended independence*.

The remainder of this paper is structured as follows. Section 2 reports our notations and definitions. Section 3 discusses the necessity of *asymmetry of desirability*. Section 4 introduces dominance axioms for preference relations on the power set. Section 5 axiomatizes the *leximax* and *leximin* extension rules. Additionally, the section introduces monotonicity and independence axioms to clarify more necessary conditions for deriving the extension rules. Finally, our conclusions are provided in Section 6.

2 Preliminary

Let X be the finite set of all alternatives with cardinality $|X|(\geq 2)$. The power set of X is denoted by $\mathscr{X}$. Let $N = \cup_{a \in X}\{n_a\}$ be the finite set of null alternatives such that 'choosing not to choose a' is equivalent to 'choosing n_a' for all $a \in X$. Furthermore, each subset of N is denoted by $N_A = \cup_{a \in A}\{n_a\}$, corresponding to each subset $A \in \mathscr{X}$. A preference relation on $X \cup N$ is assumed to be a *complete preordering* denoted by $R \in \mathscr{R}$, where $\mathscr{R}$ is the set of all preference relations on $X \cup N$. The asymmetric and symmetric components are denoted by P and I, respectively. We define (un)desirability of alternatives as follows: a is (un)desirable if and only if aPn_a (n_aPa), and neutral if and only if aIn_a for each $a \in X$. Additionally, let $\bar{R} \in \bar{\mathscr{R}}$ be a preference relation on $\mathscr{X}$, where $\bar{\mathscr{R}}$ is the set of all preference relations on $\mathscr{X}$. The asymmetric and symmetric components are denoted by $\bar{P}$ and $\bar{I}$, respectively.

Next, we discuss a method for ranking all subsets lexicographically. In Bossert [4], subsets might not be ranked if they have different cardinalities. For example, when $X = \{a, b, c\}$, $\{a, b\}$ and $\{c\}$ cannot be ranked if aIc for an agent. Several methods are used to solve this problem. Roth and Sotomayor [21] assumed a situation similar to a *college admissions problem*, and introduced the concept of empty slots.[2] Empty slots are added to each subset whose cardinality is smaller than a given quota. However, we also consider problems in more general choice theories. Furthermore, empty slots are the same with null alternatives assumed to satisfy *null-indifference*, that is, n_aIn_b for all $a, b \in X$. This restricts the scope of considerable situations, because n_aIn_b might be non-intuitive if the agent hates a and likes b. Thus, we do not assume *null-indifference* in this study.

We then introduce *transformed* subsets by adding null alternatives. For each $A \in \mathscr{X}$, let f_A: $X \to A \cup (N \setminus N_A)$ be a bijection such that $f_A(a) = a$ when $a \in A$, and $f_A(a) = n_a$ when $a \notin A$ for all $a \in X$. Let $A^* = \cup_{a \in X}\{f_A(a)\}$ be the transformed subset of A, and $\mathscr{X}^* = \cup_{A \in \mathscr{X}}\{A^*\}$ be the transformed power set of X. Furthermore, all (null) alternatives are assumed to be rearranged in descending order: for all $A^* = \{a_1^*, a_2^*, ..., a_{|X|}^*\} \in \mathscr{X}^*$, $a_i^* R a_{i+1}^*$ for all $i \in \{1, 2, ..., |X| - 1\}$. Thus, all subsets can be ranked even if they have different cardinalities by using the transformed subsets.

Finally, the *leximax* and *leximin* extension rules are defined in the following manner:

Definition 1. *Leximax extension rule* $\bar{R}_{lmax}$: $\forall A, B \in \mathscr{X}$,

$$A\bar{P}_{lmax}B \Leftrightarrow \exists i \in \{1, 2, ..., |X|\} \text{ s.t. } a_i^* P b_i^* \wedge a_j^* I b_j^* \ \forall j < i;$$

$$A\bar{I}_{lmax}B \Leftrightarrow a_i^* I b_i^* \ \forall i \in \{1, 2, ..., |X|\}.$$

Definition 2. *Leximin extension rule* $\bar{R}_{lmin}$: $\forall A, B \in \mathscr{X}$,

$$A\bar{P}_{lmin}B \Leftrightarrow \exists i \in \{1, 2, ..., |X|\} \text{ s.t. } a_i^* P b_i^* \wedge a_j^* I b_j^* \ \forall j > i;$$

$$A\bar{I}_{lmin}B \Leftrightarrow a_i^* I b_i^* \ \forall i \in \{1, 2, ..., |X|\}.$$

By using $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$, we can avoid some non-intuitive preference orders, as stated earlier in Section 1. Suppose that $X = \{a, b, c, d\}$, $aIbPn_aIn_bIn_cIn_dPcPd$, $A = \{a, c, d\}$, and $B = \{b, d\}$. Without the transformed subsets, they are forcibly ranked according to the leximax criteria as follows: $A\bar{R}B$. However, the third and second alternatives of A and B are d, and n_cPc. Thus, by transforming A and B into $A^* = \{a, n_b, c, d\}$ and $B^* = \{b, n_a, n_c, d\}$, respectively, we can obtain that $B\bar{P}_{lmax}A$.

However, $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ have a serious problem. $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ should satisfy a condition in order to provide consistent preference orders of singleton sets based on R, namely, *extensibility*.[3] This requires that a preference order of any two alternatives and one of their singleton sets are the same.

[2] There are two similar concepts: an outside option and a threshold in the preference approval voting rule (see [8]). Suppose that $X = \{a, b, c\}$. The outside option is often denoted by $\emptyset$, and $aP'bP'\emptyset P'c$ indicates that a and b are desirable and c is undesirable, where P' is the asymmetric component of R', that is, a *complete preordering* on $X \cup \emptyset$. Similarly, the threshold is denoted by $|$, and $ab|c$ is equivalent to $aP'bP'\emptyset P'c$. However, both of them are not appropriate to rank all subsets because we cannot frame the cardinalities of any two subsets using them.

[3] *Extensibility* was called an *extension rule* in related fields, such as *complete uncertainty* and *opportunity sets* (see [2]).

Extensibility: $\forall a, b \in X$, $aRb \Leftrightarrow \{a\}\bar{R}\{b\}$.

In the following example, $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ violate *extensibility*: $X = \{a, b\}$ and $n_a PaPbPn_b$. Even if aPb, $\{b\}\bar{P}_{lmax}\{a\}$ and $\{b\}\bar{P}_{lmin}\{a\}$ because $\{a\}^* = \{a, n_b\}$ and $\{b\}^* = \{n_a, b\}$. However, $n_a PaPbPn_b$ is non-intuitive because b is desirable and a is undesirable.

3 Requirements for null alternatives

To solve the above problem, we introduce properties for R and check whether they make $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ satisfy *extensibility.*

First, *consistency of desirability* requires that every desirable alternative is strictly better than every neutral or undesirable alternative, every neutral alternative is strictly better than every undesirable alternative, and any two neutral alternatives are indifferent.

Consistency of desirability: $\forall a, b \in X$, $[[aPn_a \wedge n_bRb] \vee [aIn_a \wedge n_bPb]] \Rightarrow aPb$; $[aIn_a \wedge bIn_b] \Rightarrow aIb$.

Consistency of desirability is suitable to the meaning of null alternatives, but not enough to imply *extensibility.* Suppose that $X = \{a, b\}$ and n_aPn_bPaIb. This preference order does not violate *consistency of desirability.* However, $\{b\}\bar{P}_{lmax}\{a\}$ and $\{b\}\bar{P}_{lmin}\{a\}$ even if aIb. We thus need to introduce a stronger property than *consistency of desirability.*

Now, we introduce *asymmetry of desirability.* This requires that a preference order of any two alternatives and one of their null alternatives are opposite.

Asymmetry of desirability: $\forall a, b \in X$, $aRb \Rightarrow n_bRn_a$.

From Proposition 1, *asymmetry of desirability* implies *consistency of desirability.*

Proposition 1. R satisfies *consistency of desirability* if R satisfies *asymmetry of desirability.*

Proof. Let R satisfy *asymmetry of desirability.* By way of contradiction, take any two alternatives $a, b \in X$ and assume that bRa when (i) aPn_a and n_bRb, or (ii) aIn_a and n_bPb. By *asymmetry of desirability*, bRa implies that n_aRn_b. We thus obtain aPb by *transitivity* in both cases (i) and (ii), but that is a contradiction.

Next, assume that (iii) aPb or (iv) bPa when aIn_a and bIn_b. By *asymmetry of desirability*, aPb and bPa implies n_bRn_a and n_aRn_b, respectively. Thus, we obtain bRa in case (iii) and aRb in case (iv) by *transitivity.* These results are contradictions.

Thus, R satisfies *consistency of desirability* if R satisfies *asymmetry of desirability.* □

From Lemma 1, *asymmetry of desirability* is one of the sufficient conditions for $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ to satisfy *extensibility.*

Lemma 1. $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ satisfy *extensibility* if R satisfies *asymmetry of desirability.*

Proof. Assume that R satisfies *asymmetry of desirability.* First, we prove that $\{a\}\bar{R}_{lmax}\{b\}$ implies aRb for all $a, b \in X$. Take any two alternatives $a, b \in X$ such that $\{a\}\bar{R}_{lmax}\{b\}$. The difference between $\{a\}^*$ and $\{b\}^*$ is that $a, n_b \notin \{b\}^*$ and $n_a, b \notin \{a\}^*$. By Definition 1 and *asymmetry of desirability*, $\{a\}\bar{R}_{lmax}\{b\}$ if and only if

(i) $[aRn_b \wedge n_aRb] \Rightarrow [aPn_a \vee [aIn_a \wedge n_bRb]]$;

(ii) $[aRn_b \wedge bRn_a] \Rightarrow aRb$;

(iii) $[n_bRa \wedge n_aRb] \Rightarrow [n_bPn_a \vee [n_bIn_a \wedge aRb]]$; and

(iv) $[n_bRa \wedge bRn_a] \Rightarrow [n_bPb \vee [n_bIb \wedge aRn_a]]$.

In cases (i) and (ii), aRb holds true. In cases (iii) and (iv), by way of contradiction, suppose that bPa, implying n_aRn_b by *asymmetry of desirability.* However, the assumption contradicts $n_b P n_a$ in both cases, and aRb in Case (iii). Thus, aRb holds true in all the four cases.

Next, we prove that aRb implies $\{a\}\bar{R}_{lmax}\{b\}$ for all $a, b \in X$. Take any two alternatives $a, b \in X$ such that aRb. By *asymmetry of desirability*, aRb implies $n_b R n_a$. We then obtain all the four results (i)-(iv), in other words, $\{a\}\bar{R}_{lmax}\{b\}$.

Thus, $\bar{R}_{lmax}$ satisfies *extensibility* if R satisfies *asymmetry of desirability.* Similarly, $\bar{R}_{lmin}$ satisfies *extensibility* if R satisfies *asymmetry of desirability.* □

However, *extensibility* does not imply *asymmetry of desirability.* For instance, suppose that $X = \{a, b, c\}$ and $aPn_cPn_aPn_bPbPc$ for an agent. In this case, aPb, $\{a\}\bar{P}_{lmax}\{b\}$, and $\{a\}\bar{P}_{lmin}\{b\}$, but aPb does not imply $n_b R n_a$. Thus, we should discuss the strength of *asymmetry of desirability.* First, take any two alternatives $a, b \in X$. In total, there are seventy-five preference orders of $a, b, n_a, n_b \in X \cup N$ since R is a complete preordering on $X \cup N$. In forty-five of the seventy-five orders, both $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ satisfy *extensibility.* Furthermore, in thirty-nine of the forty-five orders, R satisfies *asymmetry of desirability.* If the agent has one of the following six of the forty-five orders, $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ satisfy *extensibility*, but violate *asymmetry of desirability*: aPn_aPn_bPb, aPn_aPn_bIb, aIn_aPn_bPb, bPn_bPn_aPa, bPn_bPn_aIa, and bIn_bPn_aPa. Thus, *asymmetry of desirability* is not the necessary condition to make only $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ satisfy *extensibility.* However, there are more lexicographic extension rules, such as the *median-based* and *leximedian* ones. From the discussion, *asymmetry of desirability* might not be strong enough to consider *extensibility.* Additionally, only in thirteen of the forty-five orders, R satisfies *null-indifference.* Thus, we can relax the strong restriction of *null-indifference* and obtain consistent preference orders of singleton sets based on R by using *asymmetry of desirability.*

We then suppose that $R^\dagger$ denotes R satisfying *asymmetry of desirability.* Thus, $\bar{R}_{lmax}$ and $\bar{R}_{lmin}$ are redefined using $R^\dagger$ instead of R as follows:

Definition 3. *Leximax extension rule* $\bar{R}^\dagger_{lmax}$: $\forall A, B \in \mathscr{X}$,

$$A\bar{P}^\dagger_{lmax}B \Leftrightarrow \exists i \in \{1, 2, ..., |X|\} \text{ s.t. } a^*_i P^\dagger b^*_i \wedge a^*_j I^\dagger b^*_j \ \forall j < i;$$

$$A\bar{I}^\dagger_{lmax}B \Leftrightarrow a^*_i I^\dagger b^*_i \ \forall i \in \{1, 2, ..., |X|\}.$$

Definition 4. *Leximin extension rule* $\bar{R}^\dagger_{lmin}$: $\forall A, B \in \mathscr{X}$,

$$A\bar{P}^\dagger_{lmin}B \Leftrightarrow \exists i \in \{1, 2, ..., |X|\} \text{ s.t. } a^*_i P^\dagger b^*_i \wedge a^*_j I^\dagger b^*_j \ \forall j > i;$$

$$A\bar{I}^\dagger_{lmin}B \Leftrightarrow a^*_i I^\dagger b^*_i \ \forall i \in \{1, 2, ..., |X|\}.$$

Finally, we discuss an advantage in employing $R^\dagger$. Suppose that $\{a, b, c\}$ and $aPbPc$ for an agent. Furthermore, assume that the agent likes a and b intermediately, but hates c enormously. We then consider $\{a, c\}$ and $\{b\}$. If R satisfies *null-indifference*, $\{a, c\}^* = \{a, n_b, c\}$, $\{b\}^* = \{b, n_a, n_c\}$, and $\{a, c\}\bar{P}_{lmax}\{b\}$. However, from the setting, aPn_c is non-intuitive. This is a serious disadvantage in using the leximax criteria. If we allow that $n_c P^\dagger a$, then $\{a, c\}^* = \{a, n_b, c\}$, $\{b\}^* = \{n_c, b, n_a\}$, and we will obtain that $\{b\}\bar{P}^\dagger_{lmax}\{a, c\}$. Thus, we can express a certain degree of desirability, and adapt a part of the leximin (leximax) criteria in the leximax (leximin) extension rule by using $R^\dagger$. We thus discuss characterizations of $\bar{R}^\dagger_{lmax}$ and $\bar{R}^\dagger_{lmin}$ hereafter.

4 Axioms

Bossert [4] introduced two axioms with a fixed cardinality of subsets to characterize the group of lexicographic extension rules: *responsiveness*[4] and *neutrality*.[5] However, we characterize only the *leximax* and *leximin* extension rules separately. Thus, we employ another approach to characterize $\bar{R}^{\dagger}_{lmax}$ and $\bar{R}^{\dagger}_{lmin}$ by the following three axioms.

First, *indifference dominance* requires that $A\bar{I}B$ for all $A, B \in \mathscr{X}$ if all elements in A^* and B^* are indifferent for every rank.

Indifference dominance: $\forall A, B \in \mathscr{X}$, $[a_i^* I b_i^* \ \forall i \in \{1, 2, ..., |X|\}] \Rightarrow A\bar{I}B$.

The second (third) axioms is *prior* (*posterior*) *strict dominance*. This requires that $A\bar{P}B$ for all $A, B \in \mathscr{X}$ if each element of A^* dominates one of B^* for every rank, and there is at least one strict preference order in certain prior (posterior) parts of them. Thus, it might be said that they are partial conditions of *strict dominance*.[6]

Prior strict dominance: $\forall A, B \in \mathscr{X}$, $[\exists k \in \{1, 2, ..., |X|\}$ s.t. $[a_i^* R b_i^* \ \forall i \in \{1, 2, ..., k\}] \wedge [\exists j \in \{1, 2, ..., k\}$ s.t. $a_j^* P b_j^*]] \Rightarrow A\bar{P}B$.

Posterior strict dominance: $\forall A, B \in \mathscr{X}$, $[\exists l \in \{1, 2, ..., |X|\}$ s.t. $[a_i^* R b_i^* \ \forall i \in \{l, l+1, ..., |X|\}] \wedge [\exists j \in \{l, l+1, ..., |X|\}$ s.t. $a_j^* P b_j^*]] \Rightarrow A\bar{P}B$

5 Characterizations

First, from Lemma 2, $\bar{R}^{\dagger}_{lmax}$ and $\bar{R}^{\dagger}_{lmin}$ are *complete preorderings* on $\mathscr{X}$.

Lemma 2. $\bar{R}^{\dagger}_{lmax}$ and $\bar{R}^{\dagger}_{lmin}$ satisfy *reflexivity*, *completeness*, and *transitivity*.

Proof. From Definitions 3 and 4, $\bar{R}^{\dagger}_{lmax}$ satisfies *reflexivity* and *completeness*. Thus, we prove that $\bar{R}^{\dagger}_{lmax}$ satisfies *transitivity*.

For all $A, B, C \in \mathscr{X}$, $A\bar{P}^{\dagger}_{lmax}B$ and $B\bar{P}^{\dagger}_{lmax}C$ if and only if there exist $i, k \in \{1, 2, ..., |X|\}$ such that $a_i^* P^{\dagger} b_i^*$ and $a_j^* I^{\dagger} b_j^*$ for all $j < i$, and $b_k^* P^{\dagger} c_k^*$ and $b_l^* I^{\dagger} c_l^*$ for all $l < k$. Then, $k \leq i$ implies that $a_k^* P^{\dagger} c_k^*$ and $a_l^* I^{\dagger} c_l^*$ for all $l < k$, and $k > i$ implies that $a_i^* P^{\dagger} c_i^*$ and $a_j^* I^{\dagger} c_j^*$ for all $j < i$ from *transitivity* of $R^{\dagger}$. Thus, if $A\bar{P}^{\dagger}_{lmax}B$ and $B\bar{P}^{\dagger}_{lmax}C$, then $A\bar{P}^{\dagger}_{lmax}C$ for all $A, B, C \in \mathscr{X}$.

Next, $A\bar{P}^{\dagger}_{lmax}B$ and $B\bar{I}^{\dagger}_{lmax}C$ if and only if there exists $i \in \{1, 2, ..., |X|\}$ such that $a_i^* P^{\dagger} b_i^*$ and $a_j^* I^{\dagger} b_j^*$ for all $j < i$, and $b_k^* I^{\dagger} c_k^*$ for all $k \in \{1, 2, ..., |X|\}$. Then, $a_i^* P^{\dagger} c_i^*$ and $a_j^* I^{\dagger} c_j^*$ for all $j < i$ from *transitivity* of $R^{\dagger}$. Thus, if $A\bar{P}^{\dagger}_{lmax}B$ and $B\bar{I}^{\dagger}_{lmax}C$, $A\bar{P}^{\dagger}_{lmax}C$ for all $A, B, C \in \mathscr{X}$. Similarly, if $A\bar{I}^{\dagger}_{lmax}B$ and $B\bar{P}^{\dagger}_{lmax}C$, $A\bar{P}^{\dagger}_{lmax}C$ for all $A, B, C \in \mathscr{X}$.

Finally, $A\bar{I}^{\dagger}_{lmax}B$ and $B\bar{I}^{\dagger}_{lmax}C$ if and only if $a_i^* I^{\dagger} b_i^* I^{\dagger} c_i^*$ for all $i \in \{1, 2, ..., |X|\}$. Thus, if $A\bar{I}^{\dagger}_{lmax}B$ and $B\bar{I}^{\dagger}_{lmax}C$, then $A\bar{I}^{\dagger}_{lmax}C$ for all $A, B, C \in \mathscr{X}$.

From these results, $\bar{R}^{\dagger}_{lmax}$ satisfies *transitivity*. Similarly, $\bar{R}^{\dagger}_{lmin}$ satisfies *reflexivity*, *completeness*, and *transitivity*. □

The major result is Theorem 1 that describes the necessary and sufficient conditions to derive $\bar{R}^{\dagger}_{lmax}$ and $\bar{R}^{\dagger}_{lmin}$.

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[4] $\forall A \in \mathscr{X}_k = \{A \subseteq X \mid |A| = k\}, \forall b \in X, \forall c \in X \setminus A, bRc \Leftrightarrow A\bar{R}(A \setminus \{b\}) \cup \{c\}$.

[5] $\forall A, B \in \mathscr{X}_k, \forall \sigma : X \to X, [aRb \Leftrightarrow \sigma(a)R\sigma(b) \ \forall a \in A, \forall b \in B] \Rightarrow [A\bar{R}B \Leftrightarrow \{\sigma(a)\}_{a \in A}\bar{R}\{\sigma(b)\}_{b \in B}]$, where σ is a one-to-one mapping.

[6] $\forall A, B \in \mathscr{X}$, $[[a_i^* R b_i^* \ \forall i \in \{1, 2, ..., |X|\}] \wedge [\exists j \in \{1, 2, ..., |X|\}$ s.t. $a_j^* P b_j^*]] \Rightarrow A\bar{P}B$.

Theorem 1. $\bar{R} = \bar{R}^{\dagger}_{lmax}$ ($\bar{R} = \bar{R}^{\dagger}_{lmin}$) if and only if $R = R^{\dagger}$ and $\bar{R}$ satisfies *reflexivity*, *completeness*, *transitivity*, *indifference dominance*, and *prior* (*posterior*) *strict dominance*.

Proof. From Lemma 2 and Definition 3, $R = R^{\dagger}$, $\bar{R}^{\dagger}_{lmax}$ is a *complete preordering* on $\mathscr{X}$, and trivially satisfies *indifference dominance* and *prior strict dominance*.

We then prove that the following two propositions hold true if $R = R^{\dagger}$ and $\bar{R}$ satisfies *reflexivity*, *completeness*, *transitivity*, *indifference dominance*, and *prior strict dominance*:

(i) $A\bar{I}B \Leftrightarrow a^*_i I^{\dagger} b^*_i \ \forall i \in \{1, 2, ..., |X|\}$;

(ii) $A\bar{P}B \Leftrightarrow \exists i \in \{1, 2, ..., |X|\}$ s.t. $a^*_i P^{\dagger} b^*_i \wedge a^*_j I^{\dagger} b^*_j \ \forall j < i$.

The 'if' parts of (i) and (ii): They are trivial from $R = R^{\dagger}$, *indifference dominance*, and *prior strict dominance*.

The 'only if 'part of (i): By way of contradiction, let $A\bar{I}B$ imply the existence of some $i \in \{1, 2, ..., |X|\}$ such that $a^*_i P^{\dagger} b^*_i$ or $b^*_i P^{\dagger} a^*_i$. Suppose that i' is the argument of the minimum of i such that $a^*_i P^{\dagger} b^*_i$ or $b^*_i P^{\dagger} a^*_i$. By *prior strict dominance*, $A\bar{P}B$ if $a^*_{i'} P^{\dagger} b^*_{i'}$, and $B\bar{P}A$ if $b^*_{i'} P^{\dagger} a^*_{i'}$. They contradict $A\bar{I}B$ because $\bar{R}$ is a *complete preordering* on $\mathscr{X}$.

The 'only if' part of (ii): By way of contradiction, let $A\bar{P}B$ imply that $a^*_i I^{\dagger} b^*_i$ for all $i \in \{1, 2, ..., |X|\}$ or there exists $i \in \{1, 2, ..., |X|\}$ such that $b^*_i P^{\dagger} a^*_i$ and $a^*_j I^{\dagger} b^*_j$ for all $j < i$. Each case respectively implies that $A\bar{I}B$ or $B\bar{P}A$ by the 'if' parts of (i) and (ii). They contradict $A\bar{P}B$ since $\bar{R}$ is a *complete preordering* on $\mathscr{X}$.

Thus, $\bar{R} = \bar{R}^{\dagger}_{lmax}$ if and only if $R = R^{\dagger}$ and $\bar{R}$ is a *complete preordering* satisfying *indifference dominance* and *prior strict dominance*. Similarly, $\bar{R} = \bar{R}^{\dagger}_{lmin}$ if and only if $R = R^{\dagger}$ and $\bar{R}$ is a *complete preordering* satisfying *indifference dominance* and *posterior strict dominance*. □

Theorem 1 shows the axiomatization of $\bar{R}^{\dagger}_{lmax}$ and $\bar{R}^{\dagger}_{lmin}$. However, they seem predictable following *indifference dominance*, *prior strict dominance*, and *posterior strict dominance*. We need these critical axioms to characterize $\bar{R}^{\dagger}_{lmax}$ and $\bar{R}^{\dagger}_{lmin}$ because lexicographical comparison methods have restrictions such that we must begin to compare from the best or worst (null) alternatives and stop comparing alternatives if we find a strict preference order. Thus, it is crucial to discuss whether $\bar{R}^{\dagger}_{lmax}$ and $\bar{R}^{\dagger}_{lmin}$ satisfy other axioms. Indeed, some axioms should be satisfied by them, because we assume no compatibility of alternatives in this study. We then introduce additional axioms and show that $\bar{R}^{\dagger}_{lmax}$ and $\bar{R}^{\dagger}_{lmin}$ satisfy them.

First, *monotonicity* requires that a rank of each subset increases (decreases) by adding every (un)desirable alternative, but does not change by adding any neutral alternative. Note that its definition is based on the relationship between alternatives and their null alternatives.

Monotonicity: $\forall A \in \mathscr{X}, \forall a \in X \setminus A, \ aRn_a \Leftrightarrow A \cup \{a\}\bar{R}A$.

Next, *extended independence* requires that a preference order of any two subsets is not affected by adding a disjoint subset to both subsets.

Extended independence: $\forall A, B \in \mathscr{X}, \forall C \subseteq X \setminus (A \cup B), \ A\bar{R}B \Leftrightarrow A \cup C\bar{R}B \cup C$.

Note that *extended independence* implies *independence*[7], *extended responsiveness*[8], and other weaker related axioms. *Extended monotonicity*[9] is also defined as one of weaker

[7] $\forall a, b \in X, \forall C \subseteq X \setminus \{a, b\}, \ aRb \Leftrightarrow \{a\} \cup C\bar{R}\{b\} \cup C$.
[8] $\forall A \in \mathscr{X}, \ B \subseteq A, \forall C \subseteq X \setminus A, \ B\bar{R}C \Leftrightarrow A\bar{R}(A \setminus B) \cup C$.
[9] $\forall A \in \mathscr{X}, \forall B \subseteq X \setminus A, \ B\bar{R}\emptyset \Leftrightarrow A \cup B\bar{R}A$.

axioms than *extended independence*. However, its definition is based on the relationship between subsets and an empty set, not their null subsets. Additionally, we cannot consider another *monotonicity*[10] that is weaker than *extended monotonicity* because R is defined on $X \cup N$, which does not include an empty set. Thus, we consider *monotonicity* and *extended independence* separately.

Finally, Theorem 2 shows that $\bar{R}^{\dagger}_{lmax}$ and $\bar{R}^{\dagger}_{lmin}$ satisfy the above axioms.

Theorem 2. $\bar{R}^{\dagger}_{lmax}$ and $\bar{R}^{\dagger}_{lmin}$ satisfy *monotonicity* and *extended independence.*

Proof. First, we prove that $\bar{R}^{\dagger}_{lmax}$ satisfies *monotonicity.* Suppose that $a^*_i = n_a \in A^*$ and $B = A \cup \{a\}$. Then, $aR^{\dagger}n_a$ if and only if $a = b^*_j$, where $1 \leq j \leq i$. In this case, $b^*_k = a^*_k$ for all $k \in \{1, 2, ..., j-1, i+1, ..., |X|\}$ and $b^*_{l+1} = a^*_l$ for all $l \in \{j, 2, ..., i\}$. Thus, we obtain that $aR^{\dagger}n_a$ if and only if $aR^{\dagger}a^*_j$, which implies $B\bar{R}^{\dagger}_{lmax}A$.

Second, we prove that $\bar{R}^{\dagger}_{lmax}$ satisfies *extended independence.* Take any three subsets: $A, B \in \mathscr{X}$, and $C \subseteq X \setminus (A \cup B)$. From Definition 3, $A\bar{P}^{\dagger}_{lmax}B$ if and only if $a^*_i P^{\dagger} b^*_i$, $i \in \{1, 2, ..., |X|\}$, and $a^*_j I^{\dagger} b^*_j$ for all $j < i$. Suppose that there are k alternatives in $\tilde{C} = \{\tilde{c}_1, \tilde{c}_2, ..., \tilde{c}_k\} \subseteq C$ such that $a^*_i P^{\dagger} \tilde{c}_1 R^{\dagger} \tilde{c}_2 R^{\dagger} \cdots R^{\dagger} \tilde{c}_k P^{\dagger} b^*_i$. Then, $k = 0$ implies that $A \cup C\bar{P}^{\dagger}_{lmax}B \cup C$ because $a^*_i P^{\dagger} b^*_i$ and *transitivity* of $R^{\dagger}$. Furthermore, $k \geq 1$ also implies that $A \cup C\bar{P}^{\dagger}_{lmax}B \cup C$ because $a^*_i P^{\dagger} \tilde{c}_1$ and *transitivity* of $R^{\dagger}$. From these results, we obtain that $A\bar{P}^{\dagger}_{lmax}B$ if and only if $A \cup C\bar{P}^{\dagger}_{lmax}B \cup C$. Next, from Definition 3, $A\bar{I}^{\dagger}_{lmax}B$ if and only if $a^*_i I^{\dagger} b^*_i$ for all $i \in \{1, 2, ..., |X|\}$. In this case, the positions of $c_k \in C$ in $(A \cup C)^*$ and $(B \cup C)^*$ have to be the same for all $k \in \{1, 2, ..., |C|\}$. Finally, we obtain that $A\bar{I}^{\dagger}_{lmax}B$ if and only if $A \cup C\bar{I}^{\dagger}_{lmax}B \cup C$.

Similarly, $\bar{R}^{\dagger}_{lmin}$ satisfies *monotonicity* and *extended independence.* □

6 Conclusion

In this paper, we introduce null alternatives to frame the cardinalities of all subsets and rank them. Unlike empty slots, strict preference orders of null alternatives are allowed. According to the above framework, we define the *leximax* and *leximin* extension rules on $\mathscr{X}$.

However, if there is no requirement for null alternatives, a *complete preordering* R and the extension rules do not satisfy *extensibility.* Thus, we employ $R^{\dagger}$ on $X \cup N$, which is a *complete preordering* satisfying *asymmetry of desirability*, to obtain consistent preference orders of singleton sets based on ones of alternatives.

We then axiomatize the extension rules as follows: A preference relation $\bar{R}$ on $\mathscr{X}$ is the *leximax* extension rule if and only if R is equal to $R^{\dagger}$ and $\bar{R}$ is a *complete preordering* satisfying *indifference dominance* and *prior strict dominance.* Furthermore, a preference relation $\bar{R}$ on $\mathscr{X}$ is the *leximin* extension rule if and only if R is equal to $R^{\dagger}$ and $\bar{R}$ is a *complete preordering* satisfying *indifference dominance* and *posterior strict dominance.*

Additionally, we find that the *leximax* and *leximin* extension rules satisfy *monotonicity* and *extended independence*, which should be satisfied when there is no compatibility of alternatives.

Lastly, by using null alternatives, we can more generally apply the *leximax* and *leximin* extension rules on the power set to various fields of choice theories when we must use ordinal extension rules on the power set, and consider a certain degree of desirability.

[10] $\forall a \in X$, $\forall B \subseteq X \setminus \{a\}$, $aR'\emptyset \Leftrightarrow \{a\} \cup B\bar{R}B$, where R' is a preference relation on $X \cup N \cup \{\emptyset\}$.

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